A variational model for gradient-based video editing. Supplementary material

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1 Introduction

This document serves as a supplementary material to our submitted (IJCV) paper entitled: A variational model for gradient-based video editing. We first recall Appendix B of [10], the Euler-Lagrange equation, in Section 2 since it will used in other subsequent discussions. Then in Section 3 we discuss the Functional analytic framework and the existence of minima for our proposed energy. Section 4 provides a discussion on uniqueness of minima for the proposed energy, and Section 5 provides some remarks on the existence and uniqueness in the discrete case. We then give an extension to the DSCD discussion (Section 5 of [10]) in Section 6. Finally, Section 7 provides pseudo-code implementing the proposed method.

2 The Euler-Lagrange equation

Throughout the rest of the paper, we assume that O is a subset of $\Omega^T = \Omega \times [0, T]$ with Lipschitz boundary. Then the unit normal is defined almost everywhere on ∂O with respect to the Hausdorff measure $\mathcal{H}^2$ (surface measure) on ∂O. Let us denote by $\boldsymbol{\nu}^O = (\boldsymbol{\nu}^O_{\boldsymbol{x}}, \nu^O_t)$ the outer unit normal to ∂O (a vector in the unit sphere of $\mathbb{R}^3$) and $\boldsymbol{\nu}^{O_t}$ the outer unit normal to ∂O_t (a vector in the unit circle of $\mathbb{R}^2$).

Let us compute the Euler-Lagrange equations associated to the energy

$$E_{\boldsymbol{\kappa},\lambda}(u) = \int_O \left(\frac{1}{2} \|\boldsymbol{\kappa}(\boldsymbol{x}, t)\, \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u(\boldsymbol{x}, t)\|^2 + \frac{\lambda}{p} \|\boldsymbol{\nabla} u(\boldsymbol{x}, t)\|^p \right) \mathrm{d}\boldsymbol{x}\, \mathrm{d}t, \tag{1}$$

where $\lambda \geq 0$ and $p = 1, 2$. For that, assume that $u : O \to \mathbb{R}$ is a minimum of $E_{\boldsymbol{\kappa},\lambda}$. To compute the Euler-Lagrange equations, we consider a perturbation $\bar{u}$ such that $E_{\boldsymbol{\kappa},\lambda}(\bar{u}) < \infty$. Since u is a minimum of $E_{\boldsymbol{\kappa},\lambda}$ we have

$$\begin{aligned} \lim_{\epsilon \to 0+} \frac{E_{\boldsymbol{\kappa},\lambda}(u + \epsilon \bar{u}) - E_{\boldsymbol{\kappa},\lambda}(u)}{\epsilon} &= \int_O \boldsymbol{\kappa} \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u \cdot \boldsymbol{\kappa} \boldsymbol{\nabla} \partial_{\boldsymbol{v}} \bar{u} \, d\boldsymbol{x} dt \\ &+ \lambda \int_O \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \bar{u} \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = 0, \end{aligned}$$

where, when $\lambda > 0$ and $p = 1$, $\boldsymbol{\xi} : O \to \mathbb{R}^2$ is a measurable vector field such that $\|\boldsymbol{\xi}\|_\infty \leq 1$, $\boldsymbol{\xi} \cdot \boldsymbol{\nabla} u = |\boldsymbol{\nabla} u|$, and the arguments $(\boldsymbol{x}, t)$ of the functions are omitted for simplicity. If $\lambda > 0$ and $p = 2$, then $\boldsymbol{\xi} = \boldsymbol{\nabla} u$. Integrating by parts we have

$$\begin{aligned} 0 &= \int_O \boldsymbol{\kappa} \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u \cdot \boldsymbol{\kappa} \boldsymbol{\nabla} \partial_{\boldsymbol{v}} \bar{u} \, d\boldsymbol{x} dt + \lambda \int_O \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \bar{u} \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\ &= \int_O \partial^*_{\boldsymbol{v}} \boldsymbol{\nabla}^* (\boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u) \bar{u} \, d\boldsymbol{x} dt + \lambda \int_O \boldsymbol{\nabla}^* \boldsymbol{\xi} \, \bar{u} \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \end{aligned}$$

$$+\int_{\partial O} \boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u)(\nu_t^O + \boldsymbol{v}\cdot \boldsymbol{\nu}_{\boldsymbol{x}}^O)\bar{u}\, d\mathcal{H}^2 + \lambda \int_0^T \int_{\partial O_t} \boldsymbol{\xi}\cdot \boldsymbol{\nu}^{O_t} \bar{u} d\mathcal{H}^1 dt$$
$$+\int_0^T \int_{\partial O_t} \boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u \cdot \boldsymbol{\nu}^{O_t} \partial_{\boldsymbol{v}} \bar{u}\, d\mathcal{H}^1 dt,$$

where $d\mathcal{H}^2$, resp. $d\mathcal{H}^1$, denotes the surface measure in ∂O, resp. the length measure in ∂O_t. We have denoted by $\boldsymbol{\nabla}^*$ (resp. $\partial_{\boldsymbol{v}}^*$) the adjoint operator, that is $\boldsymbol{\nabla}^* b = -\mathrm{div}\, b$ for any vector field $b : O \to \mathbb{R}^2$ (resp. $\partial_{\boldsymbol{v}}^* \psi = -\frac{\partial \psi}{\partial t} - \mathrm{div}\,(\boldsymbol{v}\psi)$, for any function $\psi : O \to \mathbb{R}$). By taking test functions that vanish in a neighborhood of the boundary we have $\bar{u} = 0$, $\partial_{\boldsymbol{v}} \bar{u} = 0$ on ∂O and we deduce that

$$\partial_{\boldsymbol{v}}^* \boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u) + \lambda \boldsymbol{\nabla}^* \boldsymbol{\xi} = 0 \quad \text{in } O.$$

Introducing this in the above expressions we get

$$\begin{aligned}&\int_{\partial O} \boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u)(\nu_t^O + \boldsymbol{v}\cdot \boldsymbol{\nu}_{\boldsymbol{x}}^O)\bar{u}\, d\mathcal{H}^2 \\ &+\lambda \int_0^T \int_{\partial O_t} \boldsymbol{\xi}\cdot \boldsymbol{\nu}^{O_t} \bar{u} d\mathcal{H}^1 dt \\ &+ \int_0^T \int_{\partial O_t} \boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u \cdot \boldsymbol{\nu}^{O_t} \partial_{\boldsymbol{v}} \bar{u}\, d\mathcal{H}^1 dt = 0\end{aligned} \tag{2}$$

and this holds for any admissible perturbation $\bar{u}$ that will be clarified below.

Let us discuss the boundary conditions that can be specified for the problem. We use the definition and notations given in [10], Section 3.1. A set of natural boundary conditions are those for which the identity (2) holds. Let us discuss the possible choices.

Dirichlet boundary conditions. Dirichlet boundary conditions for u can be specified on a given set $A \subset \partial O$ if $\lambda > 0$ or on a subset $A \subset \partial O \setminus \partial O_{\mathrm{tang}}$ if $\lambda = 0$. Namely we can specify

$$u(\boldsymbol{x}, t) = u_0(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in A. \tag{3}$$

If u satisfies (3) and we take test functions $\bar{u}$ such that $\bar{u} = 0$ on A, then $u + \epsilon \bar{u}$ satisfies (3) and the first and second integrals in (2) vanishes on A.

Observe that, since ∂O is Lipschitz,

$$\{(\boldsymbol{x}, t) : \boldsymbol{x} \in \partial O_t, t \in (0, T)\} = \partial O_{\mathrm{tang}} \cup \partial O_{\mathrm{obli}},$$

where strictly speaking this equality holds modulo null sets with respect to the surface measure.

Specifying $\partial_{\boldsymbol{v}} u$ on the boundary. We can specify $\partial_{\boldsymbol{v}} u$ on a given subset of $\{(\boldsymbol{x}, t) : \boldsymbol{x} \in \partial O_t, t \in (0, T)\}$. Namely we can specify

$$\partial_{\boldsymbol{v}} u(\boldsymbol{x}, t) = g_0(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in B \subset \partial O_{\mathrm{tang}} \cup \partial O_{\mathrm{obli}}. \tag{4}$$

If u satisfies (4) and we take test functions $\bar{u}$ such that $\partial_{\boldsymbol{v}} \bar{u} = 0$ on $B \subset \partial O_{\mathrm{tang}} \cup \partial O_{\mathrm{obli}}$, then $u + \epsilon \bar{u}$ satisfies (4) and the third integral in (2) vanishes on B.

Specifying other boundary conditions. We can specify the boundary condition at $(\boldsymbol{x}, t) \in A' \subset \partial O$

$$\boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2 \boldsymbol{\nabla} \partial_{\boldsymbol{v}} u) \boldsymbol{\nu}^O \cdot (\boldsymbol{v}, 1) + \lambda \boldsymbol{\xi} \cdot \boldsymbol{\nu}^{O_t} = 0 \tag{5}$$

with the convention that $\boldsymbol{\xi}\cdot\boldsymbol{\nu}^{O_t}=0$ if $(\boldsymbol{x},t)\in\partial O_{\mathrm{vert}}\cup O_0\cup O_T$. Then the sum of the first and second integrals in (2) vanishes on A'.

Notice that if $\lambda=0$, (5) reduces to

$$\boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u)\boldsymbol{\nu}^O\cdot(\boldsymbol{v},1)=0 \tag{6}$$

and is trivially satisfied if $(x,t)\in\partial O_{\mathrm{tang}}$ since in that case $\boldsymbol{\nu}^O\cdot(\boldsymbol{v},1)=0$. That is, this gives no boundary condition at points $(x,t)\in\partial O_{\mathrm{tang}}$. Thus, when $\lambda=0$ we can only impose (6) on subsets $A'\subset\partial O\setminus\partial O_{\mathrm{tang}}$.

If $\lambda>0$, we can impose (5) on any subset $A'\subset\partial O$, understanding that it reduces to

$$\boldsymbol{\xi}\cdot\boldsymbol{\nu}^{O_t}=0. \tag{7}$$

Specifying $\boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u\cdot\boldsymbol{\nu}^{O_t}=0$ *on the boundary.* We can specify the boundary condition at $(\boldsymbol{x},t)\in B'\subset\partial O_{\mathrm{tang}}\cup\partial O_{\mathrm{obli}}$

$$\boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u\cdot\boldsymbol{\nu}^{O_t}=0.$$

Then the second integral in (2) vanishes on B'.

Depending on the problem we choose a set of boundary conditions. The only requirements are that

$$A\cup A'=\partial O\quad\text{if }\lambda>0\text{, or }A\cup A'=\partial O\setminus\partial O_{\mathrm{tang}}\quad\text{if }\lambda=0,$$

and

$$B\cup B'=\partial O_{\mathrm{tang}}\cup\partial O_{\mathrm{obli}}.$$

This implies that the identity (2) holds.

Boundary conditions for the one-lid setting. In the context of the one-lid problem, we choose the set of boundary conditions

$$u(\boldsymbol{x},0)=u_0(\boldsymbol{x},0),\qquad \boldsymbol{x}\in O_0, \tag{8}$$

$$u(\boldsymbol{x},t)=u_0(\boldsymbol{x},t),\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{vert}}, \tag{9}$$

$$\partial_{\boldsymbol{v}}u(\boldsymbol{x},t)=g_0(\boldsymbol{x},t)\;,\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{tang}}\setminus\partial\Omega^T, \tag{10}$$

$$\begin{array}{l}u(\boldsymbol{x},t)=u_0(\boldsymbol{x},t)\\ \partial_{\boldsymbol{v}}u(\boldsymbol{x},t)=g_0(\boldsymbol{x},t)\end{array},\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{obli}}\setminus\partial\Omega^T, \tag{11}$$

to which, when $\lambda>0$, we add

$$u(\boldsymbol{x},t)=u_0(\boldsymbol{x},t)\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{tang}}\setminus\partial\Omega^T, \tag{12}$$

where the videos u_0 and g_0 are given. Notice that the boundary condition (12) is interpreted classically if $p=2$ and it has to be interpreted in a relaxed sense if $p=1$. This is discussed with more detail in Section 3.

The boundary conditions on the rest of ∂O are

$$\boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u)(\boldsymbol{x},t)=0,\qquad \boldsymbol{x}\in O_T, \tag{13}$$

$$\begin{array}{l}\lambda\boldsymbol{\xi}\cdot\boldsymbol{\nu}^{O_t}(\boldsymbol{x},t)=0\\ \boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u(\boldsymbol{x},t)\cdot\boldsymbol{\nu}^{O_t}(\boldsymbol{x},t)=0\end{array},\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{tang}}\cap\partial\Omega^T, \tag{14}$$

$$\begin{array}{l}\boldsymbol{\nabla}^*(\boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u)(\boldsymbol{x},t)+\lambda\boldsymbol{\xi}\cdot\boldsymbol{\nu}^{O_t}(\boldsymbol{x},t)=0\\ \boldsymbol{\kappa}^2\boldsymbol{\nabla}\partial_{\boldsymbol{v}}u(\boldsymbol{x},t)\cdot\boldsymbol{\nu}^{O_t}(\boldsymbol{x},t)=0\end{array},\qquad (\boldsymbol{x},t)\in\partial O_{\mathrm{obli}}\cap\partial\Omega^T. \tag{15}$$

Boundary conditions for the two-lid setting. They are given by (8),(9),(10),(11),(14),(15), and (13) is replaced by

$$u(\boldsymbol{x}, T) = u_0(\boldsymbol{x}, T) \quad \text{in } O_T. \tag{16}$$

Let us observe that the boundary conditions (8),(9),(10), (11), and (16) in the two lid-case, are specified in the set of admissible functions in which $E_{\boldsymbol{\kappa},\lambda}$ will be minimized.

Remark. Under some assumptions on the vector field $\boldsymbol{v}$, we can prove existence and uniqueness of minima of $E_{\boldsymbol{\kappa},\lambda}$ in a suitable class of functions (the functional space where the energy is finite and permits to incorporate boundary conditions). In particular, this shows that the boundary conditions are sufficient to determine the solution. This is discussed in detail in Section 3.

3 The functional analytic framework and existence of minima of $E_{\kappa,\lambda}$

The study of minima of $E_{\kappa,\lambda}$ requires to define suitable functional spaces. For that we assume that $v \in L^\infty(\Omega \times (0,T); \mathbb{R}^2)$ with $\operatorname{div}_x v \in L^2(\Omega \times (0,T))$. We also assume that κ is a diagonal matrix with entries in $L^\infty(O)$ and $\kappa \geq \alpha I$, where I is the identity matrix and $\alpha > 0$. To fix ideas, we will consider the case $p = 1$. The case $p = 2$ is similar and more simple.

If Q is an open subset of $\mathbb{R}^N$ with Lipschitz boundary, we denote by $W^{1,2}(Q)$ the set of functions $w \in L^2(Q)$ such that $\nabla w \in L^2(Q)$. We denote by $BV(Q)$ the space of functions of bounded variation in Q. We refer to [5] for the definition and properties of BV functions.

Recall that if $w \in BV(Q) \cap L^2(Q)$ and $z \in L^\infty(Q; \mathbb{R}^2)$ is such that $\operatorname{div} z \in L^2(Q)$, then the distribution defined by

$$\int_Q z \cdot Dw\phi := -\int_Q w \operatorname{div} z \, \phi \, \mathrm{d}x - \int_Q wz \cdot \nabla\phi \, \mathrm{d}x,$$

where ϕ is a smooth test function with compact support in Q, is a Radon measure in Q such that

$$\int_Q |z \cdot Dw| \leq \|z\|_\infty \int_Q |Dw|. \tag{17}$$

The normal trace $z \cdot \nu^Q$ of z in ∂Q is well defined and the integration by parts formula holds

$$\int_Q z \cdot Dw + \int_Q w \operatorname{div} z = \int_{\partial Q} z \cdot \nu^Q w \, d\mathcal{H}^{N-1}, \tag{18}$$

where $w \in BV(Q) \cap L^2(Q)$, $\nu^Q(x)$ denotes the outer unit formal to ∂Q at $x \in \partial Q$ and $\mathcal{H}^{N-1}$ is the $N-1$-dimensional Hausdorff measure. We refer to [6] for details.

We assume that $u \in L^1_w(0,T; BV(O_t))$, that is $u : (0,T) \in BV(O_t)$ is weakly measurable, i.e. $t \in (0,T) \to u(t) \in BV(O_t)$ such that $u \in L^1(O)$ and $u \in (0,T) \to \int_{O_t} \varphi \cdot Du$ is a measurable map for any $\varphi \in C^1(O)$ with compact support in O. Then $\partial_v u = (\partial_t + v \cdot \partial_x)u$ is a distribution in O.

Notice that the energy $E_{\kappa,\lambda}$ is defined for all $u \in L^1_w(0,T; BV(O_t))$ such that $\partial_v u \in L^2(O)$ and $\nabla_x \partial_v u \in L^2(O)$. We also assume that these functions satisfy the boundary conditions (8),(9),(10),(11), and (16) in the two-lid case. Let us denote this set of functions by $\mathcal{A}$.

The boundary conditions in ∂O_{tang} have to be considered in a relaxed form. For that, we consider the energy

$$E^b_{\kappa,\lambda}(u) = E_{\kappa,\lambda}(u) + \lambda \int_{\partial O \setminus \partial\Omega^T} |u(x,t) - u_0(x,t)| \, d\mathcal{H}^2$$

defined on the class of admissible functions $\mathcal{A}$. With this the boundary integral is defined only in $\partial O_{\rm tang} \setminus \partial\Omega^T$. Then the boundary condition (12) is written as

$$\xi \cdot \nu^{O_t} \in \text{sign}(u_0(x,t) - u(x,t)) \quad (x,t) \in \partial O_{\rm tang} \setminus \partial\Omega^T. \tag{19}$$

Assume that for almost any $t \in (0,T)$ $\partial O_t \setminus \partial\Omega$ is not an $\mathcal{H}^1$ null set. We need this to use Poincaré's inequality (see [11]) in the proof of next Proposition.

Proposition 1. *Let $\lambda > 0$. There exits a minimum of $E^b_{\kappa,\lambda}$ in $\mathcal{A}$.*

Proof. Let u_n be a minimizing sequence. For almost any $t \in (0,T)$, $\partial_v u_n(t) \in W^{1,2}(O_t)$ for every n. By Poincaré's inequality [11] we have

$$\begin{aligned}\int_{O_t} |\partial_v u_n(t)|^2 \mathrm{d}x \quad &\leq \quad C_1 \int_{O_t} |\nabla_x \partial_v u_n(t)|^2 \mathrm{d}x \\ &+ \quad C_2 \int_{\partial O_t \setminus \partial\Omega} |g_0(x,t)|^2 \, d\mathcal{H}^1.\end{aligned}$$

Integrating it in $(0,T)$ and using that $E_{\kappa,\lambda}(u_n)$ is bounded we deduce that $\partial_v u_n$ is bounded in $L^2(O)$.

Now,

$$\partial_t u_n = \partial_v u_n - v \nabla_x u_n.$$

Using our assumptions on v, the fact that $u_n(t) \in BV(O_t)$ for all n, and (17) we have that $v\nabla_x u_n$ is a Radon measure in O and

$$\int_0^T \int_{O_t} |v \nabla_x u_n| \leq \|v\|_\infty \int_0^T \int_{O_t} |\nabla_x u_n| \leq \frac{\|v\|_\infty}{\lambda} E_{\kappa,\lambda}(u_n).$$

Then $\partial_t u_n$ are Radon measures and their total mass is uniformly bounded in n. Since also $\nabla_x u_n$ are Radon measures and their total mass is uniformly bounded in n, then u_n is uniformly bounded in $BV(O)$. We may extract a subsequence converging in $L^1(O)$ to a function $u \in L^1(O)$. Then $\partial_v u \in L^2(O)$ and

$$E^b_{\kappa,\lambda}(u) \leq \liminf_n E^b_{\kappa,\lambda}(u_n).$$

To prove that u is a minimum of $E_{\kappa,\lambda}$ we need to prove that u satisfies the boundary conditions (8), (9), (10), (11). Since $\{\nabla_x \partial_v u_n(t)\}_n$ is bounded in $L^2(O_t)$ for almost any $t \in (0,T)$, the boundary $\partial_v u(x,t) = g_0(x,t)$ are satisfied on $(\partial O_{\rm tang} \cup \partial O_{\rm obli}) \setminus \partial\Omega^T$.

Let us prove that u satisfies the Dirichlet boundary conditions given in (8), (9), (11). Let $\psi \in C^1(\overline{O})$. Since $u_n \in BV(O)$, using Green's formula (18) we have

$$\int_O \partial_v u_n \psi \, dxdt = \int_O u_n \partial_v^* \psi \, dxdt + \int_{\partial O} (\nu_t^O + v \cdot \nu_x^O) u_0 \psi \, d\mathcal{H}^2.$$

Since $\partial_v u_n \to \partial_v u$ weakly in $L^2(O)$ and $u_n \to u$ in $L^1(O)$, letting $n \to \infty$ we obtain

$$\int_O \partial_v u \psi \, dxdt = \int_O u \partial_v^* \psi \, dxdt + \int_{\partial O} (\nu_t^O + v \cdot \nu_x^O) u_0 \psi \, d\mathcal{H}^2. \tag{20}$$

This implies that $u(x,t) = u_0(x,t)$ in (8), (9), (11). $\square$

Remark. *Let us give a more classical point of view on the Dirichlet boundary conditions for u out of the tangential boundary. In that context, to prove that u satisfies the Dirichlet boundary conditions requires some additional assumptions on the vector field v. Let us define the incoming (resp. outgoing) boundary, that we denote by $\partial_{\rm in} O$ (resp. $\partial_{\rm out} O$), as the set of points $(x,t) \in \partial O$ such that $\nu_t^O + v \cdot \nu_x^O < 0$ (resp > 0). Notice that O_0 is part of the incoming boundary. Assume that $Z \subset \partial_{\rm in} O$ is such that for any $(\bar{x},\bar{t}) \in Z$ we have that $v \in L^1([\bar{t},\bar{t}+\delta], W^{1,\infty}(V_{\bar{x}}))$ for some $\delta > 0$ and a neighborhood of $\bar{x}$. Then we have a unique solution of the equation $X_t(t,\bar{x}) = v(X(t,\bar{x}),t)$, $t \in [\bar{t},\bar{t}+\delta]$, such that $X(\bar{t},\bar{x}) = \bar{x}$. Since $c_n = \partial_v u_n \in L^2(O)$, then we may write*

$$u_n(X(t,\bar{x}),t) = u_0(\bar{x},\bar{t}) + \int_{\bar{t}}^t c_n(X(s,\bar{x}),s)\,\mathrm{d}s. \tag{21}$$

By passing to the limit we have that

$$u(X(t,\bar{x}),t) = u_0(\bar{x},\bar{t}) + \int_{\bar{t}}^t c(X(s,\bar{x}),s)\,\mathrm{d}s. \tag{22}$$

holds a.e. $(\bar{x},\bar{t}) \in Z$ where $c \in L^2(O)$. Thus $u(\bar{x},\bar{t}) = u_0(\bar{x},\bar{t})$ on Z.

The same argument can be repeated for the outgoing boundary. This time we assume that $Z \subset \partial_{\rm out} O$ is such that for any $(\bar{x},\bar{t}) \in Z$ we have that $v \in L^1([\bar{t}-\delta,\bar{t}], W^{1,\infty}(V_{\bar{x}}))$ for some $\delta > 0$ and a neighborhood of $\bar{x}$. We deduce that $u(\bar{x},\bar{t}) = u_0(\bar{x},\bar{t})$ on Z.

In particular u satisfies the Dirichlet boundary conditions in $\partial O_{\rm vert} \cup (\partial O_{\rm obliq} \setminus \partial\Omega^T)$ if $v \in L^1(0,T;\, W^{1,\infty}(\Omega))$.

In case that v does not satisfy the local Lipschitz condition on incoming and outgoing boundary points, we cannot guarantee that the Dirichlet boundary conditions for u are satisfied in the classical sense and we consider them in the relaxed form. We can impose them on the admissible class of functions out of the tangential boundary $\partial O_{\rm tang} \setminus \partial\Omega^T$, penalizing their deviation on $\partial O_{\rm tang} \setminus \partial\Omega^T$ in the energy. We could also impose all of them by penalization in the energy. In that case, we require that admissible functions satisfy only the boundary conditions for $\partial_v u$.

The case $\lambda = 0$. Let us assume that the vector field v satisfies

$$v \in L^1(0,T;W^{1,1}(\Omega;\mathbb{R}^2)) \cap L^1(0,T;L^\infty(\Omega;\mathbb{R}^2)), \tag{23}$$

$$\operatorname{div} v \in L^1(0,T;L^\infty(\Omega)). \tag{24}$$

Those assumptions replace the assumptions on v that we did at the beginning of this Section. By extending $v(\cdot,t)$ by parity and then by periodicity to $\mathbb{R}^2$, we may assume that v is the restriction to $\Omega \times (0,T)$ of a vector field

$$v \in L^1(0,T;W^{1,1}_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)) \cap L^1(0,T;L^\infty(\mathbb{R}^2;\mathbb{R}^2)), \tag{25}$$

$$\operatorname{div} v \in L^1(0,T;L^\infty(\mathbb{R}^2)). \tag{26}$$

Those are the assumptions under which the generalized DiPerna-Lions theory of transport equations holds [8]. These results have been extended to vector fields

$$v \in L^1_w(0,T;SBD_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)) \cap L^1(0,T;L^\infty(\mathbb{R}^2;\mathbb{R}^2))$$

satisfying (26) by Ambrosio-Crippa-Maniglia [4]. We have denoted by $SBD_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ the space of vector fields $b = (b_1,b_2)$ in $L^1_{\rm loc}(\mathbb{R}^2,\mathbb{R}^2)$ such that $\frac{\partial b_1}{\partial x_2} + \frac{\partial b_2}{\partial x_1}$ is a Radon measure in $\mathbb{R}^2$ with no Cantor part. The case where

$$v \in L^1_w(0,T;BV_{\rm loc}(\mathbb{R}^2;\mathbb{R}^2)) \cap L^1(0,T;L^\infty(\mathbb{R}^2;\mathbb{R}^2))$$

and satisfies (26) has been considered in [1] (see also [3]). To fix ideas, assume that DiPerna-Lions assumptions hold.

Proposition 2. *Assume that (25) and (26) hold. Let $M > 0$. There exits a minimum of $E_{\kappa,0}$ in $\mathcal{A} \cap \{u \in L^\infty(O) : |u| \le M\}$.*

Imposing that $|u| \le M$ for some $M > 0$ is not a restrictive assumption for images, since they are bounded by the maximum intensity (usually 255).

Proof. Let us give a sketch of the proof. Let u_n be a minimizing sequence of $E_{\kappa,0}$. As in the proof of Proposition 2, $\partial_v u_n$ is bounded in $L^2(O)$. Since $|u_n| \le M$, by extracting a subsequence we may assume that $u_n \to u$ weakly* in $L^\infty(O)$ and $\partial_v u_n \to \partial_v u$ weakly in $L^2(O)$. Then $\partial_v u \in L^2(O)$ and by the lower semicontinuity of the energy we have

$$E_{\kappa,0}(u) \le \liminf_n E_{\kappa,0}(u_n).$$

To prove that u is a minimum of $E_{\kappa,0}$ we need to prove that u satisfies the boundary conditions (8), (9), (10), (11). Since $\{\nabla_x \partial_v u_n(t)\}_n$ is bounded in $L^2(O_t)$ for almost any $t \in (0,T)$, the boundary condition $\partial_v u(x,t) = g_0(x,t)$ is satisfied on $(\partial O_{\text{tang}} \cup \partial O_{\text{obli}}) \setminus \partial\Omega^T$.

Let us prove that u satisfies the Dirichlet boundary conditions given in (8), (9), (11). By our assumptions on v and the results in [6, 4], $(1,v)u$ has a trace on ∂O and the integration by parts formula (20) holds for u_n and any $\psi \in C^1(\overline{O})$. Letting $n \to \infty$, (20) holds for u and any $\psi \in C^1(\overline{O})$. The boundary conditions for u are satisfied in this weak sense. □

Remark 1. *Although the assumptions for v above are quite general, they may not be sufficient to cover real video cases, since optical flow may have discontinuities along curves and its divergence may be a Radon measure. In the continuous framework, one could compute the optical flow by imposing constraints that guarantee at least that $\operatorname{div} v$ has some integrability properties, e.g. being in L^2.*

Remark 2. *Let us comment again on the classical point of view to prove existence when $\lambda = 0$. In this case we do not assume that admissible functions are bounded by M. Assume that $\overline{O} \subset \Omega^T$, v satisfies (25) and (26), and $\partial_{\text{vert}} O \cup \partial_{\text{obli}} O = \emptyset$. Under that conditions, for almost any $x \in \Omega$ we have a unique solution of the equation $X_t(t,x) = v(X(t,x),t)$ such that $X(0,x) = x$. Assume for simplicity that all trajectories in O start at O_0 and end on O_T. In that case, we can get a bound on u_n in $L^2(O)$. Indeed, if $c_n = \partial_v u_n \in L^2(O)$, then we may write*

$$u_n(X(t,x),t) = u_0(x,0) + \int_0^t c_n(X(s,x),s)\,\mathrm{d}s. \tag{27}$$

Since v satisfies (25) and (26), the Jacobian of the map $y = X(t,x)$ is bounded and bounded away from zero [8] and from the above identity we deduce that u_n is bounded in $L^2(O)$.

As in the case $\lambda > 0$, the boundary conditions that specify $\partial_v u$ are satisfied. Also the Dirichlet boundary conditions on O_0 and O_T are satisfied.

The consideration of existence and uniqueness results of solutions of transport equations and the corresponding ordinary differential equations in bounded domains under very mild conditions leads to more deep mathematical analysis and is not the object of the present paper. We refer to [7] for a uniqueness result when the boundary of the domain is transversal to the flow. General existence results in $\mathbb{R}^N$ or in bounded domains where the flow is tangential can be found in [8, 1, 4, 2, 3].

Remark 3. *Notice that we had to assume that $\kappa \ge \alpha I$, where I is the identity matrix and $\alpha > 0$, for any $(x,t) \in O$. The above techniques can also be adapted to consider the case where $\kappa(x,t) = 0$ for $(x,t) \in \Gamma \subset O$ where Γ is a closed set of zero measure.*

We will discuss below the discrete approach to these problems.

4 On uniqueness of minima of $E_{\kappa,\lambda}$

Let us assume that the vector field satisfies assumptions (25) and (26). The proof holds both for $p = 1, 2$.

Let u_1, u_2 be two minima of $E^b_{\kappa,\lambda}$ in $\mathcal{A}$ (or of $E_{\kappa,0}$ in $\mathcal{A} \cap \{u \in L^\infty(O) : |u| \leq M\}$). If $\nabla_x \partial_v u_1 \neq \nabla_x \partial_v u_2$, since the quadratic term of the energy is strictly convex, then

$$E_{\kappa,\lambda}\left(\frac{u_1+u_2}{2}\right) < \frac{1}{2}E_{\kappa,\lambda}(u_1) + \frac{1}{2}E_{\kappa,\lambda}(u_2).$$

Since $\frac{u_1+u_2}{2} \in \mathcal{A}$, this contradicts the fact that u_1, u_2 are minima of $E_{\kappa,\lambda}$.

Let $u = u_1 - u_2$. Then $\nabla_x \partial_v u = 0$ in O and all boundary conditions (8),(9),(10),(11) (plus (16) in the two-lid case) hold with homogeneous right hand side. This implies that $\partial_v u = 0$ in O. By (20) we have that

$$\int_O u \partial_v^* \psi \, dxdt = 0 \quad \forall \psi \in C^1(\overline{O}). \tag{28}$$

Now, for any test function $\phi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ (that is, infinitely differentiable with compact support in $\mathbb{R}^2 \times (0,T)$)we consider the solution of

$$\frac{\partial \Psi}{\partial t} + \operatorname{div}(v\Psi) = \phi \quad \text{in } \mathbb{R}^2 \times (0,T),$$

with initial condition $\Psi(0) = 0$ in $\mathbb{R}^2$ [8]. Let $\rho_\epsilon(x) = \frac{1}{\epsilon^2}\rho(\frac{x}{\epsilon})$ where $\rho \in \mathcal{D}(\mathbb{R}^2)$, $\rho \geq 0$, and $\int_{\mathbb{R}^2} \rho(x)\, dx = 1$. By the regularization result in [8], we have that $\Psi_\epsilon = \rho_\epsilon * \Psi$ satisfies

$$\frac{\partial \Psi_\epsilon}{\partial t} + \operatorname{div}(v\Psi_\epsilon) = \phi + r_\epsilon \quad \text{in } \mathbb{R}^2 \times (0,T),$$

where $r_\epsilon \to 0$ in $L^1(0,T; L^1_{\text{loc}}(\mathbb{R}^2))$. By replacing ψ by Ψ_ϵ in (28) we have

$$\int_O u(\phi + r_\epsilon)\, dxdt = 0 \quad \forall \phi \in \mathcal{D}(\mathbb{R}^2 \times (0,T)). \tag{29}$$

Letting $\epsilon \to 0+$ we obtain

$$\int_O u\phi \, dxdt = 0 \quad \forall \phi \in \mathcal{D}(\mathbb{R}^2 \times (0,T)). \tag{30}$$

This implies that $u = 0$. That is, $u_1 = u_2$.

5 Remarks on existence and uniqueness in the discrete case

For the discrete discussion, we will use the same notation as in the continuous domain as we did in the paper.

Let us consider the energy (1) in the discrete case which amounts to replace the integrals in O by sums, that is

$$E^d_{\kappa,\lambda}(u) = \sum_{(x,t)\in\widetilde{O}} \|\kappa(x,t)\nabla_x \partial_v u(x,t)\|^2 + \lambda \sum_{(x,t)\in\widetilde{O}} \|\nabla_x u(x,t)\|^p,$$

where $\lambda \geq 0$ and $p = 1, 2$. The energy is defined in vectors $u \in \mathcal{X} := \mathbb{R}^{|\widetilde{O}|}$. The boundary conditions have been described in the Section entitled "Definition of the Operators" in the paper.

Assume first that $\lambda > 0$. If u_n is a minimizing sequence for $E^d_{\kappa,\lambda}$, then we have that $\nabla_x u_n$ is bounded. From the Dirichlet boundary conditions, we deduce that u_n is bounded in $\mathcal{X}$. Then we may extract subsequence converging to $u \in \mathcal{X}$ satisfying the boundary conditions. Then u is a minimum of $E^d_{\kappa,\lambda}$. This result has been obtained for any $\kappa \geq 0$.

When $\lambda = 0$, we assume that $\kappa \geq \alpha I$, $\alpha > 0$. In that case, we first observe that $\nabla_x \partial_v u_n$ is bounded. From the specification of $\partial_v u_n$ on the boundary of each O_t (out of $\partial\Omega^T$), we deduce that $\partial_v u_n$ is bounded. Getting from this the boundedness of u_n requires specifying the discretization of ∂_v. To illustrate this (our treatment will be sketchy), assume that ∂_v is discretized as the backward derivative $\partial_v^b u$. Assume that $u_n(x,t)$ is bounded in O_t uniformly in n. Then $\hat{u}_n(x + v^b(x,t+1),t)$ is also bounded, being based on bilinear interpolation of the values of $u_n(x.t)$. Since $u_n(x,t+1) = \partial_v^b u_n(x,t+1) + \hat{u}_n(x+v^b(x,t+1),t)$ we deduce that $\{u_n(x,t+1) : x \in O_{t+1}\}$ is bounded uniformly in n. Assume now that ∂_v is discretized as the forward derivative $\partial_v^f u$ and $u_n(x,t)$ is bounded in O_t uniformly in n. Then $\hat{u}_n(x + v^f(x,t), t+1) = \partial_v^f u_n(x,t) + u_n(x,t)$ and we get that $\{\hat{u}_n(x+v^f(x,t),t+1) : x \in O_t\}$ is bounded uniformly in n. The flow has to be dense enough so that, from this and bi-linear interpolation equations , we can get that $\{u_n(x,t+1) : x \in O_{t+1}\}$ is bounded uniformly in n. Clearly, this depends on the optical flow and for that reason it is convenient to use $\lambda > 0$. The same conclusions apply to the DSCD scheme.

When $\lambda > 0$ and $p = 2$, the energy is strictly convex and uniqueness follows. When $\lambda > 0$ and $p = 1$, or $\lambda = 0$, uniqueness is a more delicate issue. As in Section 4, uniqueness is reduced to prove that if $\nabla_x \partial_v u = 0$ in $\widetilde{O}$ and all boundary conditions (8),(9),(10),(11) (plus (16) in the two-lid case) hold with homogeneous right hand side, then $u = 0$. In a first step, from the specification of $\partial_v u$ on each ∂O_t we get that $\partial_v u = 0$. Getting from this that $u = 0$, we need to be able to connect by the flow v each pixel (x,t) in the interior of $\widetilde{O}$ to a boundary pixel where u is specified. To fix ideas, let us consider the case of the DSCD based on the odd-assignation. Assume that

$$h_v^{\mathrm{odd}} u(x,t) = 0. \tag{31}$$

Since $u(x,0) = 0$, the interpolation of intermediate values gives $\hat{u}(x + v^b(x,1),0) = 0$. Hence

$$u(x,1) = \hat{u}(x + v^b(x,1),0) = 0 \quad \forall x \in O_1.$$

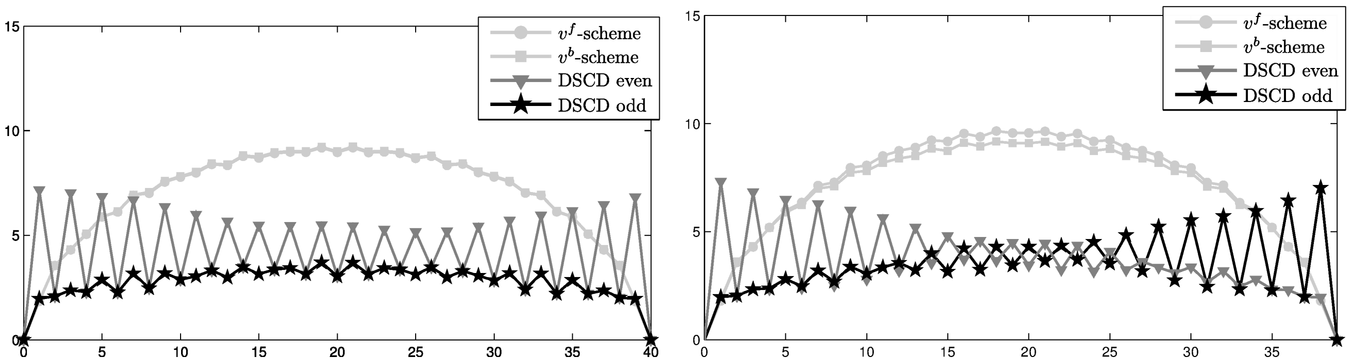
Now,

$$\hat{u}(x + v^f(x,1),2) = u(x,1) = 0 \quad \forall x \in O_1.$$

We know that $u(x,2) = 0$ $\forall x \in \partial O_2$. The important point here is that, given our interpolation model, the density of points $x + v^f(x,1)$ has to be sufficient to guarantee that $\hat{u}(x + v^f(x,1),2) = 0$ $\forall x \in O_1$ implies that $u(x,2) = 0$ $\forall x \in O_2$. By iterating this argument, we obtain that $u = 0$. This requires an information on the optical flow v that cannot be guaranteed before hand. Here, the use of conjugate gradient method can help to stabilize the numerical solution.

6 Further discussion on the DSCD

This Section complements Sections 5 and 6 in [10]. In Section 5 we recall the *Deblurring Scheme for the Convective Derivative* (DSCD) initially introduced in [9]. The idea of DSCD is to alternate between the $\boldsymbol{v}^b$ and $\boldsymbol{v}^f$ schemes, one frame each. In this way the blurring effect of the $\boldsymbol{v}^b$ scheme is cancelled out by the sharpening effect of the next $\boldsymbol{v}^f$ scheme. In Section 5.1, we analyze the particular case of a zero energy solution (*i.e.* compatible boundary conditions) with an optical flow corresponding to a constant translation. In this case the cancellation can be shown to be exact. The resulting propagation scheme, when looked each two frames, has a flat frequency response and an approximately linear phase, thus approximating an ideal shift operator (in accordance to the translational motion).



(a) Two-lids case, odd number of frames.

(b) Two-lids case, even number of frames.

Figure 1: Evolution of the root mean square error w.r.t. the ground truth corresponding to two synthetic problems with a constant translation. In (a) the sequence has 41 frames with an optical flow of $v_0 = [-0.425, 0]$ px/frame. In (b) the sequence has 40 frames with an optical flow of $v_0 = [-0.436, 0]$ px/frame.

6.1 Behaviour of even and odd DSCDs in a two-lid setting

This analysis, (as discussed in Section 5.2) holds only for the described particular case. However, it sheds light over more realistic cases, with non-compatible boundary data and more complex motions. Let us consider in particular, a slight modification of the above-mentioned example by placing a second lid.

Figure 1 shows the resulting RMSE for the *smiley* experiment shown in [10], Figures 9 and 10. Here the flow is still a translation, but we have placed two-lids, resulting in non-compatible boundary data. In this case, we have to distinguish two cases depending on the whether the number of frames in the sequence is even or odd. Figure 1(a) shows the result with 41 frames (the first lid is frame $t = 0$ whereas the second lid is frame $t = 40$). This Figure is the same as the one shown in [10], Figure 10(b). We show it here to facilitate the comparison. In Figure 1(b) we remove one frame and place the second lid at frame $t = 39$. The behaviour of the $\boldsymbol{v}^b$ and $\boldsymbol{v}^f$ is roughly the same, but there is a changes in the behaviour of the even and odd DSCDs: for the sequence with 41 frames, the even and the odd DSCDs coincide at even frames, and at odd frames, the even DSCD has a higher RMSE, associated with high frequency artifacts introduced by the sharpening step.

When the total number or frames is even (Figure 1(b)), for the first 10 frames in the sequence, the behaviour of even and odd DSCDs resembles the one in Figure 1(a): Both DSCDs coincide at even frames, and the even DSCD yields high RMSE at odd frames. However, the situation is inverted by the end of the sequence towards the second lid: the DSCDs coincide on odd frames, and it is odd DSCD the one with high RSME at even frames.

The reasons for this become clear when we write the DSCD energies in terms of the M^b and M^f interpolation filters:

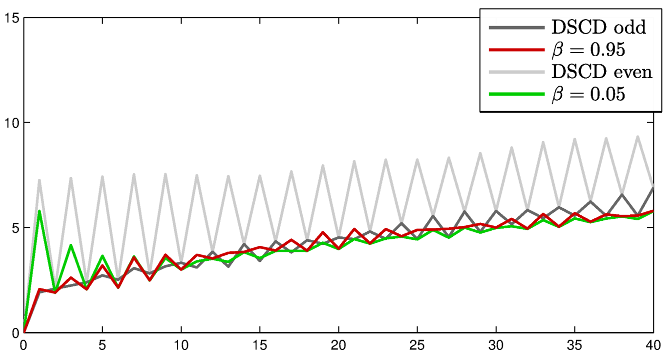
$$E_{\boldsymbol{\kappa}}^{\mathrm{odd}}(u) = \|M^b u_0(\,\cdot\,,0) - u(\,\cdot\,,1)\|^2 + \|u(\,\cdot\,,1) - M^f u(\,\cdot\,,2)\|^2 + \cdots + \begin{cases} \|M^b u(\,\cdot\,,T-1) - u_0(\,\cdot\,,T)\|^2 & \text{if } T+1 \text{ is odd,} \\ \|u(\,\cdot\,,T-1) - M^f u_0(\,\cdot\,,T)\|^2 & \text{if } T+1 \text{ is even.} \end{cases} \tag{32}$$

$$E_{\boldsymbol{\kappa}}^{\mathrm{even}}(u) = \|u_0(\,\cdot\,,0) - M^f u(\,\cdot\,,1)\|^2 + \|M^b u(\,\cdot\,,1) - u(\,\cdot\,,2)\|^2 + \cdots + \begin{cases} \|u(\,\cdot\,,T-1) - M^f u_0(\,\cdot\,,T)\|^2 & \text{if } T+1 \text{ is odd,} \\ \|M^b u(\,\cdot\,,T-1) - u_0(\,\cdot\,,T)\|^2 & \text{if } T+1 \text{ is even.} \end{cases} \tag{33}$$

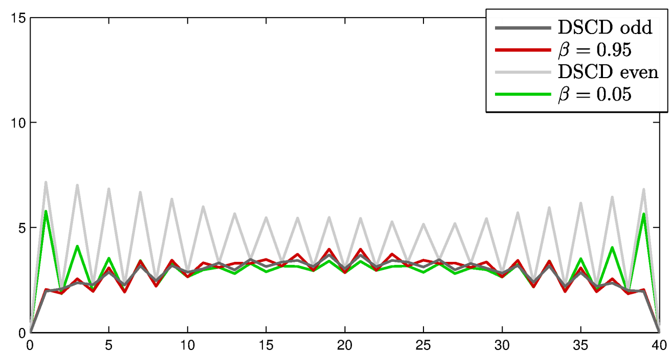
The first term determines the nature of the connection with the first lid. As discussed in [10], Section 5, the odd DSCD enforces an explicit (averaging) link between the lid and $u(\,\cdot\,,1)$. For the even DSCD, on the other hand, the link is implicit, responsible for the high RMSE erros at odd frames in close to the first lid.

The nature of the connection to the last lid depends on the parity of the total number of frames $T+1$. For an odd number of frames the connection to the last lid is of the same type as the connection to the first lid. The odd DSCD is linked to the second lid via a $\boldsymbol{v}^f$ step, which when seen in the backwards direction of propagation, is an explicit (and thus averaging) step. The even DSCD, is linked to the last lid through a term enforcing an implicit sharpening relation between $u(\cdot, T-1)$ and the lid, when seen in the backwards direction. This the reason for the symmetric behaviour of both DSCD curves in Figure 1(a).

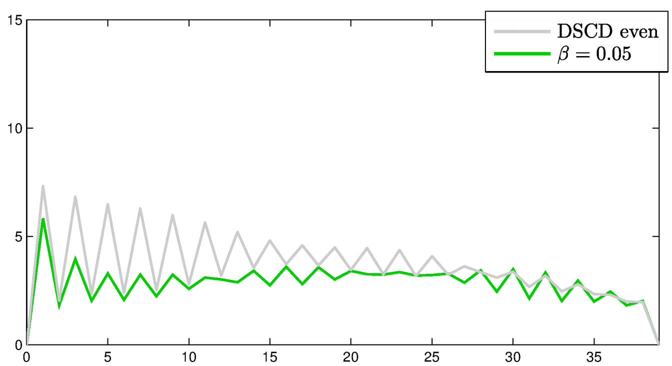
However, when the $T+1$ is even, the situation is reversed: the odd DSCD establishes an implicit link to the second lid, whereas the even DSCD is linked explicitly. This explains the exchange in the behaviour of even and odd DSCDs in Figure 1(b).



(a) One-lid case.



(b) Two-lids case, odd number of frames.



(c) Two-lids case, even number of frames.

Figure 2: Evolution of the root mean square error w.r.t. the ground truth corresponding to two synthetic problems with a constant translation. In this plots we compare the "pure" even and odd DSCDs with the combination of both according to energy E_β in Eq. 34. We consider $\beta = 0.05$ (95% of even with 5% of odd) and $\beta = 0.95$ (5% of even with 95% of odd). In plots (a) and (b) the sequence has 41 frames with an optical flow of $v_0 = [-0.425, 0]$ px/frame. In (c) the sequence has 40 frames with an optical flow of $v_0 = [-0.436, 0]$ px/frame. Here we only show the graphs for the even DSCD and to $\beta = 0.05$. The ones corresponding to the odd DSCD and to $\beta = 0.95$ correspond to a specular symmetry of the ones showed.

6.2 RMSE curves for the combination of both DSCDs

In [10] and as a way to compensate for high frequenciy artifacts introduced in the sharpening steps of both DSCDs, we define in Section 6 a new energy by a convex combination of $E_{\boldsymbol{\kappa}}^{\text{odd}}$ and $E_{\boldsymbol{\kappa}}^{\text{even}}$. Let us recall it here:

$$E_\beta = \beta E_{\boldsymbol{\kappa}}^{\text{odd}} + (1-\beta) E_{\boldsymbol{\kappa}}^{\text{even}}, \tag{34}$$

where $\beta \in [0,1]$ determines the weight of each DSCD in the combination. The even and odd DSCDs are particular cases of this energy corresponding to $\beta = 0$ and $\beta = 1$ respectively. Typically, β is set to values close to 0 or 1, to add a bit of averaging to the sharpening steps.

In [10], Figure 9 shows results obtained with $\beta = 0.05$ and $\beta = 0.95$ for the *smiley* synthetic one-lid experiment. These results show that the high frequency artifacts of the "pure" DSCDs are removed (at the expense of a mild blurring). In [10], we omitted from Figures 10(a) and 10(b) the RMSE curves (corresponding to $\beta = 0.05$ and $\beta = 0.95$) to avoid cluttering the graphs and to simplify the discussion.

For the sake of completeness, we now show these curves in Figure 2, for a one-lid and two two-lids problems, with even and odd total number of frames.

Thus, at least for the *smiley* example, the combination of both DSCDs has a favourable impact in terms of RMSE. In particular, the peaks associated with the averaging steps are highly reduced (except for the first pair of peaks after a lid).

These RMSE curves are shown to provide a quantitative performance measure complementing the presentation of the motivational example in [10], Section 5. We refer the reader to [10], Figure 9 for a qualitative assessment. In particular, let us note that the although the RMSE of the combination of both DSCDs is similar to the one of the odd DSCD, the perceived image quality of the combination of both DSCD is clearly better.

7 Implementation and pseudo-code

In this Section we explain how to build the sparse linear system for minimizing $E_{\boldsymbol{\kappa}}^{\text{odd}}(u)$ of [10, Eq. (33)]. First we rewrite the energy in an equivalent form which allows a simpler implementation, and then we present a pseudocode for building and solving the system.

Simplification of the energy. We start by observing that minimizing

$$E_{\boldsymbol{\kappa}}^{\text{odd}}(u) = \sum_{\widetilde{O}} \|\boldsymbol{\kappa}^{\text{odd}} \boldsymbol{\nabla}^{\text{odd}} h_{\boldsymbol{v}}^{\text{odd}} u\|^2,$$

w.r.t. u is equivalent to solving the constrained problem (defined over $\Omega \times \{0, 1, \cdots, T\}$):

$$\min_u \sum_{\Omega \times \{0,1,\cdots,T\}} \|\boldsymbol{\kappa}^{\text{odd}} \boldsymbol{\nabla}^{\text{odd}} h_{\boldsymbol{v}}^{\text{odd}} u\|^2 \qquad \text{s.t.} \quad u|_{O^c} = u_0,$$

where O^c denotes the complement of O. Since h_v^{odd} only keeps the forward and backward convective derivatives at odd frames, we can split the energy to expose the forward and backward terms

$$\sum_{\Omega \times \{0,1,\cdots,T\}} \|\boldsymbol{\kappa}^{\text{odd}} \boldsymbol{\nabla}^{\text{odd}} h_{\boldsymbol{v}}^{\text{odd}} u\|^2 = \sum_{\Omega \times \{0,1,\cdots,T\}} \|\boldsymbol{\kappa}^f \boldsymbol{\nabla}^f \left(O_{odd}\, \partial_{\boldsymbol{v}}^f u\right)\|^2 + \sum_{\Omega \times \{0,1,\cdots,T\}} \|\boldsymbol{\kappa}^b \boldsymbol{\nabla}^b \left(O_{odd}\, \partial_{\boldsymbol{v}}^b u\right)\|^2$$

where $O_{odd}(\boldsymbol{x}, t)$ is 1 if t is odd, and 0 otherwise. The convective derivatives $\partial_{\boldsymbol{v}}^{f,b}$ are implemented according to [10, Eq. (21),(22)]. Let us introduce the finite difference gradient $\bar{\nabla} q$ defined over the whole domain $\Omega \times \{0, 1, \cdots, T\}$

and with appropriate Neumann boundary conditions. Recalling the definitions of $\nabla^{f,b}$ and the sets $S^{f,b}$ and $\widetilde{S}^{f,b}_{e_i}$ from [10, Eq. (24) to (26)] we observe that we can re-write the gradients in simpler terms

$$\nabla^{f,b} q := \begin{bmatrix} \widetilde{S}^{f,b}_{e_1} & 0 \\ 0 & \widetilde{S}^{f,b}_{e_2} \end{bmatrix} \bar{\nabla} q, \tag{35}$$

where the S-sets are being used as indicator functions (i.e. $S(\boldsymbol{x}, t)$ is 1 if $(\boldsymbol{x}, t) \in S$, and 0 otherwise). Incorporating these definitions in the energy we get:

$$\sum_{\Omega\times\{0,1,\cdots,T\}} \left\| \boldsymbol{\kappa}^f \begin{bmatrix} \widetilde{S}^{f}_{e_1} & 0 \\ 0 & \widetilde{S}^{f}_{e_2} \end{bmatrix} \bar{\nabla} \left(O_{odd}\, \partial^f_{\boldsymbol{v}} u\right) \right\|^2 + \sum_{\Omega\times\{0,1,\cdots,T\}} \left\| \boldsymbol{\kappa}^b \begin{bmatrix} \widetilde{S}^{b}_{e_1} & 0 \\ 0 & \widetilde{S}^{b}_{e_2} \end{bmatrix} \bar{\nabla} \left(O_{odd}\, \partial^b_{\boldsymbol{v}} u\right) \right\|^2 .$$

We further simplify the energy by collapsing the tensors $\bar{\boldsymbol{\kappa}}^f := \boldsymbol{\kappa}^f \begin{bmatrix} \widetilde{S}^{f}_{e_1} & 0 \\ 0 & \widetilde{S}^{f}_{e_2} \end{bmatrix}$

$$E^{\text{odd}}_{\boldsymbol{\kappa}}(u) = \sum_{\Omega\times\{0,1,\cdots,T\}} \|\bar{\boldsymbol{\kappa}}^f\, \bar{\nabla} \left(O_{odd}\, \partial^f_{\boldsymbol{v}} u\right)\|^2 + \sum_{\Omega\times\{0,1,\cdots,T\}} \|\bar{\boldsymbol{\kappa}}^b\, \bar{\nabla} \left(O_{odd}\, \partial^b_{\boldsymbol{v}} u\right)\|^2. \tag{36}$$

Implementation and minimization. The following pseudocode explains how the linear system for solving (36) is constructed. We will use the following conventions: the monochrome input video u_0, the masks and the flow fields v^f, v^b treated as lexicographically ordered 1d vectors. However, for simplicity we use (x, t) as indices of the entries of the 1d vectors, and the notation $[A(u)](x, t)$ to refer to rows of matrices (index of (x, t)-th row of A in this case).

1. Let m be the number of pixels in $\Omega \times \{0, 1, \cdots, T\}$.
2. Generate the masks of the editing domain O, its complement O^c and the pixels in the even/odd frames O_{even} and O_{odd} respectively. Also compute the masks $S^f, \widetilde{S}^f_{e_i}, S^b, \widetilde{S}^b_{e_i}$ and the occlusion tensors $\boldsymbol{\kappa}^f, \boldsymbol{\kappa}^b$ (see [10, Section 4]). And compute the collapsed tensors $\bar{\boldsymbol{\kappa}}^{f,b} := \begin{bmatrix} \tilde{S}^{f,b}_{e_1}\kappa^{f,b}_{e_1} & 0 \\ 0 & \tilde{S}^{f,b}_{e_2}\kappa^{f,b}_{e_2} \end{bmatrix}$ (as justified above).
3. Construct the following sparse matrices (with lexicographically ordered entries):
   - $K^{f,b}$: $2m \times 2m$ binary diagonal matrices acting on gradients and implementing $\bar{\boldsymbol{\kappa}}^{f,b}$.
   - $S^{f,b}, O, O^c, O_{odd}, O_{even}$: $m \times m$ binary diagonal matrices acting on images and implementing the homonymous masks.
   - I^f_v: $m \times m$ matrix implementing the forward warping by v^f of its input $[I^f_v(u)](x, t) = \hat{u}(x + v^f(x), t+1)$, where $\hat{u}$ denotes the bi-linear or bi-cubic interpolation of u. Similarly I^b_v for the backward warping.
   - I_0: $m \times m$ identity matrix.
   - $J^f_v := I^f_v - I_0$. $m \times m$ matrix implementing $\partial^f_v \cdot$, the forward convective derivative. Similarly $J^b_v := I_0 - I^b_v$.
   - G: $2m \times m$ matrix implementing $\bar{\nabla}\cdot$, the spatial gradient $G := \left[\begin{smallmatrix} G_{e_1} \\ G_{e_2} \end{smallmatrix}\right]$, with $[G_{e_i}(u)](x, t) = u(x + e_i, t) - u(x, t)$.
4. Build the operator. For implementing (36) we write a $4m \times m$ matrix

$$A := \begin{bmatrix} K^f G\, O_{odd}\, J^f_v \\ K^b G\, O_{odd}\, J^b_v \end{bmatrix},$$

the energy will be $E^{odd}_{\boldsymbol{\kappa}}(u) = u^T A^T A u$. And for implementing $E_{\boldsymbol{\beta},\boldsymbol{\lambda}}(u)$ we write a $10m \times m$ matrix

$$A := \begin{bmatrix} \beta K^f G \, O_{odd} \, J^f_v \\ \beta K^b G \, O_{odd} \, J^b_v \\ (1-\beta) K^f G \, O_{even} \, J^f_v \\ (1-\beta) K^b G \, O_{even} \, J^b_v \\ \lambda G \end{bmatrix}.$$

5. Resolution of: $u^* = \arg\min_u \|Au\|^2$ s.t. $u|_{O^c} = u_0$.
   - Since the variables $u|_{O^c}$ are fixed we can split the variable $u = Ou + O^c u_0$ and rewrite the problem as $\|AOu + AO^c u_0\|^2$ with with homogenous constraints $u|_{O^c} = 0$. Its solution is then obtained by solving the linear system $Su = b$ where $S := OA^T AO$ and $b = -OA^T AO^c u_0$.
   - The final video is recovered as $u^* = S^{-1} b + u_0$.

Note that because of the restriction matrices O and $K^{f,b}$ the final system to solve is much smaller than $m \times m$.

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